

Modeling stock return distributions with a quantum harmonic oscillator

K. Ahn^{†‡}, M.Y. Choi^{*§}, B. Dai[†], S. Sohn[‡] and B. Yang[‡]

[†]Moon Soul Graduate School of Future Strategy, Korea Advanced Institute of Science and Technology, 34141 Daejeon, Republic of Korea

[‡]HSBS Business School, Peking University, 518055 Shenzhen, P.R. China

[§]Department of Physics and Astronomy and Center for Theoretical Physics, Seoul National University, 08826 Seoul, Republic of Korea

We propose a quantum harmonic oscillator as a model for the market force which draws a stock return from short-run fluctuations to the long-run equilibrium. Analyzing the Financial Times Stock Exchange (FTSE) All Share Index, we demonstrate that our model outperforms traditional stochastic process models, e.g., geometric Brownian motion and the Heston model, with smaller fitting errors and better goodness of fit statistics. The solution of the Schrödinger equation for the quantum harmonic oscillator shows that stock returns follow a mixed χ distribution, which describes Gaussian and non-Gaussian features of the stock return distribution. In addition, we provide an economic rationale of the physics concepts such as the eigenstate, eigenenergy, and angular frequency, which sheds light on the relationship between finance and econophysics literature.

Keywords: Fokker Plank equation; Quantum harmonic oscillator; Stock return distribution

JEL Classification: C13, D30

1. Introduction

In recent years extensive research has been devoted to investigating stock return distributions for asset pricing, risk management, and asset allocation purposes. One important model of stock price evolution is the geometric Brownian motion (GBM), which assumes that the logarithm of a stock price follows a Brownian motion with drift and results in a Gaussian distribution for log stock returns. However, empirical evidence illustrates that the distribution of stock returns has non-Gaussian properties including negative skewness and positive excess kurtosis (Ataullah *et al.* 2009). To describe the characteristics of stock return distribution better, there have been proposed many models such as the variance gamma model (Madan and Seneta 1990), Laplace distribution model (Linden 2001), and Heston model (Drăgulescu and Yakovenko 2002).

As an alternative to traditional stock return models, an increasing number of quantum models have also been applied to study stochastic dynamics of stock prices (Ye and Huang 2008, Ataullah *et al.* 2009, Zhang and Huang 2010, Pedram 2012, Cotfas 2013, Meng *et al.* 2014, 2015). Some of these studies successfully capture non-Gaussian properties of the stock return distribution: For instance, Ataullah *et al.* (2009) regarded stock returns as a particle evolving in a finite square potential well and Meng *et al.* (2014) analyzed the Chinese stock index by means of quantum Brownian motion. The advantage of such quantum models over the traditional stock return models lies in the

*Corresponding author. Email: mychoi@snu.ac.kr

incorporation of market conditions on the stock returns, which is captured by the potential term in the Hamiltonian. Given these features of quantum models, however, few provide the rationale of choosing potential wells and the economic explanation of physics concepts.

Besides deviations from the Gaussian distribution, another consensus on stock return behavior is that relatively high or low stock returns will dissipate as investors exploit excess profits. This implies that there exists a market force which draws a stock return from short-run fluctuations to long-run equilibrium, which is supported by the evidence of mean reversion in stock returns (Balvers et al. 2000). Among potentials in quantum models, we consider the harmonic potential to which any potential approximates near the equilibrium captures this market force. Specifically, the harmonic potential determines a location-dependent drift term in the stochastic process, meaning that the restoring force to the equilibrium is proportional to the displacement. Consequently, our model provides a revisit to the Ornstein-Uhlenbeck (OU) process in the quantum context. We then solve the Fokker-Planck (FP) equation for the probability density function (PDF) of stock returns. The solution is a linear combination of the eigenfunctions of the Hamiltonian. Our model outperforms the traditional models, such as GBM and the Heston model, in fitting the empirical distribution of FTSE All Share Index returns.

Some studies investigate derivative pricing applying eigenfunction expansion (Davydov and Linetsky 2003, Boyarchenko and Levendorskii 2007). For instance, Davydov and Linetsky (2003) unbundle contingent claim into portfolios of primitive securities called eigensecurities. The pricing problem reduces to the regular Sturm-Liouville (SL) problem, and the solutions to such a problem form a complete orthonormal basis in the Hilbert space. In this paper, we present the application of eigenfunction expansion in a physics framework. By introducing a quantum harmonic oscillator, our FP equation can be converted to a time-independent Schrödinger equation, which plays the similar role of SL equation in Davydov and Linetsky (2003). Furthermore, unlike Davydov and Linetsky (2003) who simply use eigenfunctions as a mathematical tool, we provide an interpretation of the eigenspectrum in economics and finance contexts. For example, eigenstates can be regarded as different uncertainty regimes in finance, and the eigenenergies of each state as the degree of investors' collective trading activities, i.e., the pressure on stock prices. The difference in eigenenergies between two states is the barrier for the stock to overcome and to transmit to higher uncertainty regimes. We also explain the relationship among holding periods, speed of price adjustment, and return volatility in line with finance literature. These economic implications could help us gain a deeper insight into the essentials of stock return behavior.

This paper consists of five sections: In section 2 we propose the quantum harmonic oscillator model. Section 3 presents the methodology and data. In section 4 we mainly explain economic implications of the physics concept. Finally, section 5 concludes the paper. Two traditional stochastic models are discussed in the Appendix.

2. Quantum harmonic oscillator

Let us consider a standard Wiener process W_t and the following stochastic differential equation

$$dx = v(x, t)dt + \sigma(x, t)dW_t. \quad (1)$$

Introducing the PDF $\rho(x, t)$ of the random variable x at time t , we obtain the FP equation from Eq. (1):

$$\frac{\partial}{\partial t}\rho(x, t) = \frac{\partial^2}{\partial x^2} [D(x, t)\rho(x, t)] + \frac{\partial}{\partial x} \left[\rho(x, t) \frac{\partial V(x, t)}{\partial x} \right], \quad (2)$$

where $D(x, t) \equiv \sigma^2(x, t)/2$ is the diffusion coefficient and $V(x, t)$ is the external potential determining the drift term according to $v(x, t) \equiv -\partial V(x, t)/\partial x$. In the simple case of constant D and

time-independent potential $V(x)$, Eq. (2) can be expressed in terms of the FP operator:

$$\frac{\partial}{\partial t}\rho(x, t) = \left[\frac{\partial^2 V}{\partial x^2} + \frac{\partial V}{\partial x} \frac{\partial}{\partial x} + D \frac{\partial^2}{\partial x^2} \right] \rho(x, t) \equiv \hat{L}\rho(x, t). \quad (3)$$

Note that the operator \hat{L} is non-hermitian because of the first derivative. This can be remedied by transforming the FP equation in Eq. (3) to a Schrödinger equation with a hermitian Hamiltonian. To achieve this, we introduce a new function (Petroni *et al.* 1998):

$$\phi(x, t) \equiv \frac{\rho(x, t)}{\sqrt{\rho_s(x)}},$$

where $\rho_s(x)$ is the stationary solution of Eq. (2) (Putz 2016):

$$\rho_s(x) = \frac{1}{C} e^{-V(x)/D} \quad (4)$$

with the normalization constant $C \equiv \int_{-\infty}^{+\infty} dx e^{-V(x)/D}$. Then the FP operator in Eq. (3) leads to $\hat{L}\rho(x, t) = -\sqrt{\rho_s(x)}\hat{H}\phi(x, t)$, where the hermitian Hamiltonian operator \hat{H} is given by

$$\hat{H} = -\frac{1}{2} \frac{\partial^2 V}{\partial x^2} + \frac{1}{4D} \left(\frac{\partial V}{\partial x} \right)^2 - D \frac{\partial^2}{\partial x^2}.$$

The FP equation is now expressed as the time-dependent Schrödinger equation in imaginary time $\tau = -i\hbar t$:

$$i\hbar \frac{\partial}{\partial \tau} \phi(x, \tau) = \hat{H} \phi(x, \tau) = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \phi(x, \tau) + U(x) \phi(x, \tau) \quad (5)$$

with the mass $m \equiv \hbar^2/2D$ and effective potential (Krylov 2002)

$$U(x) \equiv -\frac{1}{2} \frac{\partial^2 V(x)}{\partial x^2} + \frac{1}{4D} \left[\frac{\partial V(x)}{\partial x} \right]^2.$$

The general solution of Eq. (5) takes the form

$$\phi(x, \tau) = \sum_{n=0}^{\infty} A_n \phi_n(x) \exp\left(-\frac{i}{\hbar} E_n \tau\right),$$

where A_n is the amplitude of the (normalized) solution $\phi_n(x)$ of the time-independent Schrödinger equation: $\hat{H}\phi_n(x) = E_n\phi_n(x)$ with eigenenergy E_n . The solution of the FP equation thus reads

$$\rho(x, t) = \sqrt{\rho_s(x)} \sum_{n=0}^{\infty} A_n \phi_n(x) \exp(-E_n t),$$

where the amplitude is determined by the initial PDF $\rho(x, 0)$ according to

$$A_n = \int_{-\infty}^{\infty} dx \phi_n^*(x) [\rho_s(x)]^{-1/2} \rho(x, 0).$$

Note that Eq. (5) describes the dynamics of a particle of mass m in the potential $U(x)$. The Taylor expansion of $U(x)$ around the equilibrium point x_0 , defined by $dU/dx|_{x_0} = 0$, reads

$$U(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \left. \frac{d^n U}{dx^n} \right|_{x_0} (x - x_0)^n.$$

In case that deviations from the equilibrium are small, we may neglect terms of higher-order in $x - x_0$ and write

$$U(x) = U(0) + \frac{1}{2} k x^2 \quad (6)$$

with $k \equiv d^2 U/dx^2|_0$, where we have taken $x_0 \equiv 0$ without loss of generality. In this way, $U(x)$ is described by a harmonic potential and the system reduces to a harmonic oscillator.

Classically, $F \equiv -dU/dx = -kx$ corresponds to the restoring force, which pushes the particle out of the equilibrium position back to the equilibrium one. Further, $\omega \equiv \sqrt{k/m}$ gives the angular frequency of the harmonic oscillator. A higher value of ω leads to faster adjustment to the long-run equilibrium from short-run fluctuations. Here the mass m represents the firm-specific characteristics that determine the speed of price adjustment, such as the market capitalization and trading volumes (Meng *et al.* 2015, Zhang and Huang 2010, Chordia and Swaminathan 2000). In consequence, under common market conditions described by k , different speeds of price adjustment can be observed across firms (Damodaran 1993). It is of interest that in classical mechanics, the particle position is given by a deterministic function of time t , governed by Newton's law of motion; this is analogous to the behavior of stock prices with zero volatility that yields a deterministic trajectory. In reality, however, stock price evolution appears indeed random. Such randomness can conveniently be taken into account by quantum noise inherent in the formulation of quantum mechanics and the probabilistic description based on quantum mechanics is useful to probe the "random evolution" of stock prices.

We thus consider small deviations from the equilibrium and resort to the quantum harmonic oscillator, which is described by Eq. (5) with the effective potential in the form of Eq. (6). Specifically, taking the harmonic potential $V(x) = \gamma x^2$, we obtain the effective potential in the harmonic form as well:

$$U(x) = -\gamma + \frac{1}{2} m \omega^2 x^2$$

with $\gamma = \omega \sqrt{mD/2}$. It is well known that the n th eigenfunction of the harmonic oscillator is given by

$$\phi_n(x) = \frac{1}{\sqrt{2^n n!}} \left(\frac{m\omega}{\pi \hbar} \right)^{1/4} H_n \left(\sqrt{\frac{m\omega}{\hbar}} x \right) \exp \left(-\frac{m\omega}{2\hbar} x^2 \right)$$

with the corresponding eigenenergy

$$E_n = \left(n + \frac{1}{2} \right) \hbar \omega - \gamma = n \hbar \omega,$$

where H_n is the n th Hermite polynomial.¹ A few low-lying eigenfunctions $\phi_n(x)$ for $n \leq 5$ are shown in Figure 1.

¹The first few Hermite polynomials are given by $H_0(u) = 1$, $H_1(u) = 2u$, $H_2(u) = 4u^2 - 2$, $H_3(u) = 8u^3 - 12u$, etc.

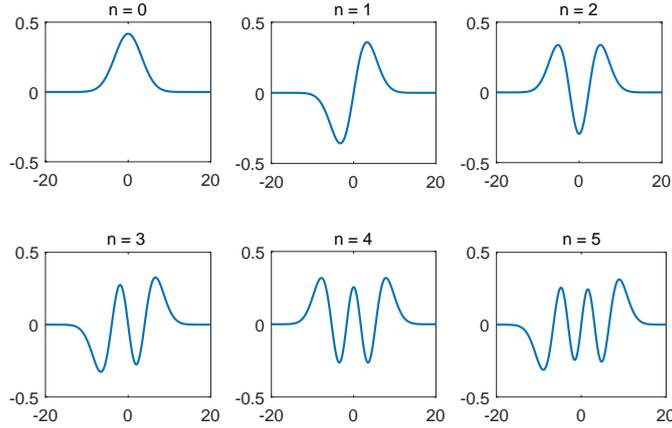


Figure 1. Low-lying eigenfunctions of a quantum harmonic oscillator

With Eq. (4) given by

$$\rho_s(x) = \sqrt{\frac{m\omega}{\pi\hbar}} \exp\left(-\frac{m\omega}{\hbar}x^2\right),$$

we finally obtain the solution of the FP equation

$$\rho(x, t) = \sum_{n=0}^{\infty} \frac{A_n}{\sqrt{2^n n!}} \sqrt{\frac{m\omega}{\pi\hbar}} \exp(-E_n t) H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right) \exp\left(-\frac{m\omega}{\hbar}x^2\right). \quad (7)$$

Note that this solution takes the form of a mixed χ distribution:

$$\rho(x, t) = \sum_{n=0}^{\infty} C_n(t) \rho_n(x) \quad (8)$$

with $C_n(t) = (A_n/\sqrt{2^n n!})\sqrt{m\omega/\pi\hbar}e^{-E_n t}$ and $\rho_n(x) = H_n(\sqrt{m\omega/\hbar}x)e^{-(m\omega/\hbar)x^2}$. For example, we have $\rho_0(x) \propto f(\sqrt{2m\omega/\hbar}x; 1)$, $\rho_1(x) \propto f(\sqrt{2m\omega/\hbar}x; 2)$, $\rho_2(x) \propto f(\sqrt{2m\omega/\hbar}x; 3) - f(\sqrt{2m\omega/\hbar}x; 1)$, etc. with $f(x; k) = \frac{2^{1-k/2}}{\Gamma(k/2)} x^{k-1} e^{-x^2/2}$, where k is the degree of freedom and $\Gamma(z)$ is the Gamma function.

Since $E_n = n\hbar\omega$, terms of $n \geq 1$ in the summation of Eqs. (7) or (8) decays exponentially with time t . In particular, in the limit $t \rightarrow \infty$, only the ground-state ($n = 0$) term survives. As a result, the initial memory gets lost and returns follow, regardless of the initial distribution, exactly the Gaussian distribution, in which case the model reduces to GBM for the price process. At finite time t , on the other hand, the incorporation of excited states ($n \geq 1$) increases thickness of the tail, displaying leptokurtic. The mixture of even and odd states leads to asymmetry of the distribution, which captures skewness. Therefore the excited states serve to capture the stylized facts of stock returns, i.e., skewness and kurtosis. Note here that except at very short time t , higher-order terms become very small. We thus need to consider only a few eigenstates of small n , which makes it feasible to manage Eq. (7) for the fitting purpose.

3. Empirical analysis

3.1. Data

We calibrate the models using the daily FTSE All Share Index from 15 November, 2007 to 21 September, 2014. This period is of interest, as it covers the global financial crisis, the European sovereign debt crisis, and post-recession periods. The data have been collected from the Bloomberg database. The daily, weekly, and monthly continuously compounded returns

$$R_t^{ann} = \frac{252.5}{\tau} \ln \left(\frac{S_{t+\tau}}{S_t} \right)$$

are annualized for $\tau = 1, 5$ and 20 trading days, respectively.

Table 1 summarizes the statistics of stock returns, which are leptokurtic with negative skewness. It is thus manifested that returns do not follow a Gaussian distribution.

Table 1. Summary statistics of stock returns for different holding periods (τ)

τ	No. of obs.	Mean	Std.	Skewness	Excess kurtosis
1	1746	0.0508	3.3070	-0.1540	7.0114
5	1742	0.0549	1.3824	-0.6503	5.6287
20	1727	0.0534	0.6545	-1.2350	4.0501

3.2. Estimation

We estimate the parameters of the quantum model by minimizing the Cramér-von Mises goodness of fit statistic (Ataullah *et al.* 2009):

$$T_3(\Theta) = \frac{1}{12M} + \sum_{j=1}^M \left[F(r_j; \Theta) - \frac{j-1/2}{M} \right]^2,$$

where $r_j \equiv R_j - \bar{R}$ is the j th ordered centered return with \bar{R} being the historical average return used as a proxy for the long-run equilibrium, M is the total number of observations and $F(r_j; \Theta)$ is the accumulated area under the probability density below the j th ordered centered return for given parameter set Θ .

As remarked in section 2, the amplitudes of the 6th and higher eigenstates are rather small and negligible. We thus consider only the first five eigenstates ($0 \leq n \leq 4$). There are in total six undetermined parameters, which are the amplitudes of five eigenstates, C_n for $n = 0, 1, 2, 3$, and 4, and an additional one $m\omega$ appearing as a whole in the PDF given by Eq. (8). These six parameters are subject to one constraint, which is the normalization condition for $\rho(x)$. Table 2 gives the Cramér-von Mises goodness of fit statistic T_3 . The null hypothesis of the Cramér-von goodness of fit test is that data come from given distribution F . If T_3 is larger than the tabulated critical value, the null hypothesis can be rejected.

We also estimate the parameters of GBM and the Heston model (see Appendix). Estimated parameters are presented in Table 3.

To compare the fitting results, we use the Cramér goodness of fit statistic, which is calculated in the following way: First, we determine the 5th, 10th, and up to 100th percentiles of returns. We then find the actual number N_{5i} of empirical returns, falling between the $5(i-1)$ th and the $5i$ th

Table 2. Cramér-von Mises goodness of fit statistics

τ	T_3
1	0.0440
5	0.0343
20	0.0377
Critical value (1%)	0.7435
Critical value (5%)	0.4614
Critical value (10%)	0.3473

Table 3. Parameter estimates

Models	Parameters	τ		
		1	5	20
GBM	μ	0.0704	0.0738	0.0703
	σ^2	0.0433	0.0378	0.0339
Heston	θ	1.558×10^{-4}	1.513×10^{-4}	1.383×10^{-4}
Quantum	C_0	0.1708	0.3658	0.7506
	C_1	0.0035	0.0157	0.0646
	C_2	0.0208	0.0299	0.0517
	C_3	-0.0021	-0.0086	-0.0361
	C_4	0.0047	0.0064	0.0096
	$m\omega$	9.666×10^{-36}	4.434×10^{-35}	1.866×10^{-34}

percentiles, and evaluate

$$T_0 = \sum_{i=1}^{20} \frac{(N_{5i} - E_{5i})^2}{E_{5i}},$$

where E_{5i} is the expected number of returns falling between the $5(i-1)$ th and $5i$ th percentiles under the distribution F . The goodness of fit statistic T_0 is asymptotically distributed as a χ^2 variate with $(n-k-1)$ degrees of freedom, where n is the number of percentiles and k the number of parameters estimated from the data.

The null hypothesis of the Cramér goodness of fit test is that data come from the distribution F . If T_0 is larger than the critical value, the null hypothesis can be rejected. Note that the degree of freedom is 17 for GBM, 18 for the Heston model, and 14 for our quantum model.²

Table 4. Cramér goodness of fit tests

τ	GBM	p-value	Heston	p-value	Quantum	p-value
1	236.96	0.0000	43.35	0.0007	22.82	0.0633
5	138.05	0.0000	57.95	0.0004	15.63	0.3362
20	186.76	0.0000	148.79	0.0000	20.99	0.1090

According to Table 4, we reject the null hypothesis that the data come from the distribution of GBM or the Heston model since all the p-values are smaller than 0.01. In case of the quantum harmonic oscillator, the p-value of daily data is larger than 0.05 while the p-values of weekly and monthly data are well above 0.1. Thus one may not reject the null hypothesis that data come from the distribution of the quantum harmonic oscillator model.

²There are six parameters in the quantum model. However, since they should satisfy one constraint, the number of free parameters reduces to five.

4. Discussion

We plot in Figure 2 the fitted PDF of each model along with the empirical distribution, and in Figure 3 the fitting error of each model. They demonstrate that our quantum model results in the smallest fitting errors, thus visually confirming that our model provides a more adequate description of the empirical distribution. Specifically, GBM severely understates and overstates the probability density of log returns around zero and in the moderate positive and negative ranges, respectively; the Heston model exaggerates the probability density of small positive or negative returns, and this exaggeration becomes worse as the holding period increases. In contrast, the fitting error of the quantum model remains small in any range of log returns and is affected little by the holding period. Together with the goodness of fit statistics shown in Table 4, we conclude that the quantum approach outperforms the traditional stock return models.

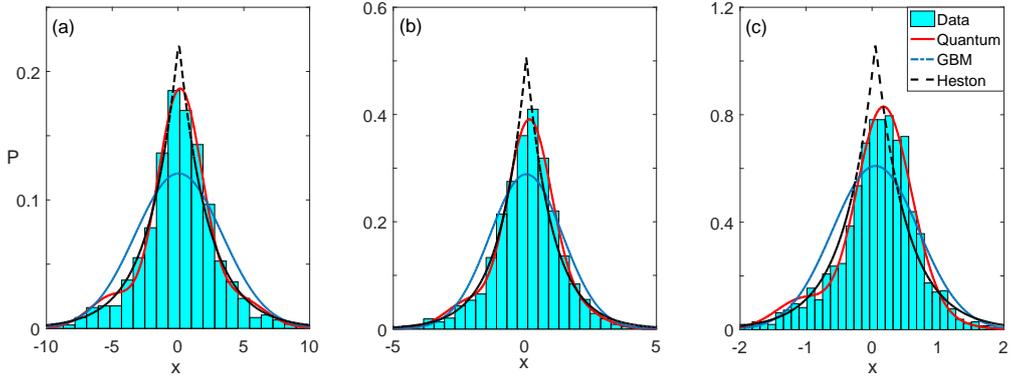


Figure 2. PDF of log returns x in GBM (blue), the Heston model (black), and our quantum model (red), for the holding period $\tau =$ (a) 1, (b) 5, and (c) 20. The empirical data are also plotted (histogram).

The sources of such good fit are (i) the incorporation of the market uncertainty, which was modeled purely as a random walk in the traditional stock return models, through the properties of wave functions and (ii) the consideration of the market force which draws short-run fluctuations to the long-run equilibrium through the quantum harmonic oscillator.

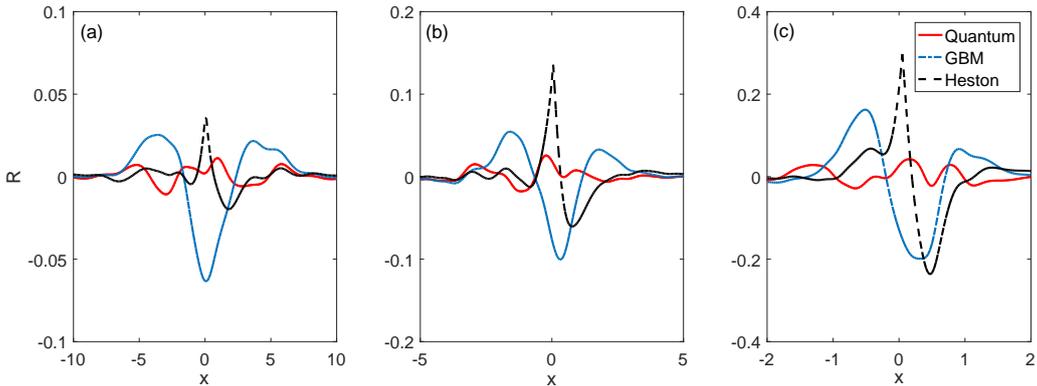


Figure 3. Residual plots corresponding to Figure 2.

As addressed in section 2, the solution of the Schrödinger equation is expressed as a linear combination of eigenfunctions corresponding to discrete eigenstates. Eigenstates are associated with a set of discrete values of physical quantities, such as energy levels. For the n th eigenstate, the energy E_n and variance σ_n^2 are given by linear functions of the quantum number n : $E_n = n\hbar\omega$

and $\sigma_n^2 = (2n+1)(\hbar/2m\omega)$. The most stable eigenstate is the ground state ($n = 0$), with the lowest energy and variance. In order for a particle to transit to an excited state at a higher energy level, it must absorb energy enough to make a quantum jump to that excited state, which also has a larger variance.

If we interpret the variance of the quantum state as the level of market uncertainty, i.e., the stock market volatility, and the energy as the degree of investors' collective trading activities, i.e., the pressure on stock prices, then the quantum model is commensurate with the study in existing finance literature. For instance, it was argued that accumulated price pressure exceeding some threshold can induce large price movements and that the stock market volatility could exhibit a quantum change (Abreu and Brunnermeier 2003), which is consistent with the properties of the quantum model. In particular, higher market uncertainty corresponds to a higher energy level. Therefore the market tends to limit high volatility by putting a high energy threshold on it.

Note that the probability for the particle in an eigenstate is proportional to the square of the amplitude of that eigenstate. Accordingly, $P_n \equiv N^{-1}|C_n|^2$ with the normalization factor $N \equiv \sum_{k=0}^4 |C_k|^2$ represents the probability of a stock return residing in the n th eigenstate. Table 5 presents the probability P_n computed for $n = 0$ to 4. It is shown that the ground state ($n = 0$), following the Gaussian distribution, has a probability higher than 90 percent regardless of the holding period. This indicates that the stock market tends to be bounded mostly at the smallest uncertainty level. In other words, it has a small possibility to be in the eigenstates with higher volatility, which is compatible with the stylized fact that there is an equilibrium level to which volatility will eventually return in the long run (Engle and Patton 2001).

Although the ground state takes the largest probability, we still observe nuances of probabilities across different holding periods. In Table 5, as the holding period increases, the probabilities of odd states also increase while those of even states decrease. This makes it possible to explain the properties of moments, shown in Table 1: The presence of odd states accounts for the asymmetry of the distribution (see Figure 1). A more asymmetric distribution with a larger skewness (longer holding period) would thus have larger probabilities of odd states. On the other hand, even states, which are symmetric, contribute to the fat tail and lead to a higher kurtosis. Therefore we find that returns in longer holding periods have lower probabilities of even states and are less leptokurtic with lower excess kurtosis.

Table 5. Probabilities of five low-lying eigenstates

τ	P_0	P_1	P_2	P_3	P_4
1	0.9842	0.0004	0.0145	0.0001	0.0008
5	0.9907	0.0018	0.0066	0.0005	0.0003
20	0.9856	0.0073	0.0047	0.0023	0.0001

The disparity across different holding periods has its origin in the parameter ω which characterizes the harmonic oscillator. Since m is interpreted as the market capitalization (or firm-specific characteristics in general), it is persistent for different holding periods of one stock index. Therefore, in line with the evidence in Table 3, the parameter ω increases as the stock is held for longer period. Since ω is the angular frequency measuring the rate of oscillations around the equilibrium, we interpret ω as the speed of mean reversion of stock returns. During short holding periods when investors aim to speculate in stocks, greater information disparity and the resulting bias lead to price overreaction, thus retarding the price reversion process and leading to a lower speed of mean reversion, and vice versa for long holding periods. On the other hand, a lower mean reversion speed in stock returns results in a more volatile distribution (Cox *et al.* 1985). This helps to explain the negative relationship between the holding period and stock return volatility, which keeps parallel with Atkins and Dyl (1997).

5. Conclusion

Considering that the market always draws back the stock return from short-run fluctuations to the long-run equilibrium, we have proposed a model based on a quantum harmonic oscillator and demonstrated empirical evidence with the FTSE All Share Index. It has been found that our model based on a quantum harmonic oscillator outperforms the traditional stochastic process models, leading to smaller fitting errors and better goodness of fit statistics. The incorporation of market uncertainty through the properties of wave functions is one of the sources of such excellent performance. The model shows that stock returns follow a mixed χ distribution, among which the ground state is Gaussian and the excited states contribute to non-Gaussian features. We also provide the economic rationale of physics concepts: While the eigenstates correspond to uncertainty regimes, the difference in the eigenenergy between two states represents the barrier between the two regimes.

One can think of extensions of our approach to various other problems. An example is to apply it to international comparison, e.g., the U.S. vs China, which gives an insight on the difference between the two markets. There exists 10% return limitation in the Chinese stock market, in which case the infinite square well might serve as a more proper potential. Another extension involves application to returns of different portfolios, e.g., large vs small, value vs growth, etc. Other than the stock returns, it is also feasible to model the interest rate through the quantum approach and apply it to the bond market. Further, our model can also be applied to risk management, e.g., computing Value at Risk based on the PDF of the quantum harmonic oscillator and comparing it with that from historical simulations or extreme-value theory.

Acknowledgments

This work was supported by Peking University HSBC Business School through Bridge Trust Asset Management Research Fund (K.A. and S.S.), by KAIST through High Risk High Return Project (K.A.) and by National Research Foundation through the Basic Science Research Program (Grant No. 2016R1D1A1A09917318 M.Y.C.).

References

- Abreu, D. and Brunnermeier, M., Bubbles and crashes. *Econometrica*, 2003, **71(1)**, 173-204.
- Ataullah, A., Davidson, I. and Tippett, M., A wave function for stock market returns. *Physica A: Statistical Mechanics and its Applications*, 2009, **388(4)**, 455-461.
- Atkins, A. and Dyl, E., Transactions costs and holding periods for common stocks. *Journal of Finance*, 1997, **52(1)**, 309-325.
- Balvers, R., Wu, Y. and Gilliland, E., Mean reversion across national stock markets and parametric contrarian investment strategies. *Journal of Finance*, 2000, **55(2)**, 745-772.
- Boyarchenko, N. and Levendorskiĭ, S., The Eigenfunction Expansion Method in Multi-Factor Quadratic Term Structure Models. *Mathematical finance*, 2007, **17(4)**, 503-539.
- Chordia, T. and Swaminathan, B., Trading volume and cross-autocorrelations in stock returns. *Journal of Finance*, 2000, **55(2)**, 913-935.
- Cotfas, A., A finite-dimensional quantum model for the stock market. *Physica A: Statistical Mechanics and its Applications*, 2013, **392(2)**, 371-380.
- Cox, J., Ingersoll, J. and Ross, S., A theory of the term structure of interest rates. *Econometrica*, 1985, 385-407.
- Damodaran, A., A simple measure of price adjustment coefficients. *Journal of Finance*, 1993, **48(1)**, 387-400.
- Davydov, D. and Linetsky, V., Pricing options on scalar diffusions: an eigenfunction expansion approach. *Operations research*, 2003, **51(2)**, 185-209.
- Drăgulescu, A. and Yakovenko, V., Probability distribution of returns in the Heston model with stochastic volatility. *Quantitative Finance*, 2002, **2(6)**, 443-453.

- Engle, R. and Patton, A., What good is a volatility model. *Quantitative Finance*, 2001, **1(2)**, 237-245.
- Krylov, G., On solvable potentials for one dimensional Schrödinger and fokker-planck equations. *Nonlinear Phenom Complex Syst*, 2002, **3**, 336-343.
- Linden, M., A model for stock return distribution. *International Journal of Finance and Economics*, 2001, **6(2)**, 159-169.
- Madan, D. and Seneta, E., The variance gamma model for share market returns. *Journal of Business*, 1990, 511-524.
- Meng, X., Zhang, J. and Guo, H., Quantum Brownian motion model for stock markets. *Physica A: Statistical Mechanics and its Applications*, 2014, **452**, 281-288.
- Meng, X., Zhang, J., Xu, J. and Guo, H., Quantum spatial-periodic harmonic model for daily price-limited stock markets. *Physica A: Statistical Mechanics and its Applications*, 2015, **438**, 154-160.
- Pedram, P., The minimal length uncertainty and the quantum model for the stock market. *Physica A: Statistical Mechanics and its Applications*, 2012, **391(5)**, 2100-2105.
- Petroni, C., De Martino, S. and De Siena, S., Exact solutions of fokker-planck equations associated to quantum wave functions. *Physics Letters A*, 1998, **245(1-2)**, 1-10.
- Putz, M. V., *Quantum Nanochemistry, Volume two: Quantum Atoms and Periodicity*, 2016 (CRC Press: Boca Riton, FL).
- Scott, D., On optimal and data-based histograms. *Biometrika*, 1979, **66(3)**, 605-610.
- Silva, C., Prange, R. and Yakovenko, V., Exponential distribution of financial returns at mesoscopic time lags: a new stylized fact. *Physica A: Statistical Mechanics and its Applications*, 2004, **344(1-2)**, 227-235.
- Ye, C. and Huang, J., Non-classical oscillator model for persistent fluctuations in stock markets. *Physica A: Statistical Mechanics and its Applications*, 2008, **387(5)**, 1255-1263.
- Zhang, C. and Huang, L., A quantum model for the stock market. *Physica A: Statistical Mechanics and its Applications*, 2010, **389(24)**, 5769-5775.

Appendix

The GBM is a stochastic process for stock price S_t governed by the following stochastic differential equation:

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

where μ is the drift, σ the volatility, and W_t a standard Wiener process. Then the continuously compounded return $x \equiv \ln(S_\tau/S_0)$ during holding period τ follows the Gaussian distribution

$$x \sim N \left[\left(\mu - \frac{\sigma^2}{2} \right) \tau, \sigma^2 \tau \right].$$

In the Heston model the stochastic variance σ_t^2 follows a Cox, Ingersoll, and Ross process (Cox *et al.* 1985), defined by

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma_t S_t dW_{1,t} \\ d\sigma_t^2 &= -\gamma(\sigma_t^2 - \theta) dt + \kappa \sigma_t dW_{2,t} \\ d\langle W_{1,t}, W_{2,t} \rangle &= \rho dt, \end{aligned}$$

where θ is the mean reversion level for the variance, γ is the mean reversion speed for the variance, κ is the variance noise, and ρ is the correlation coefficient between two Wiener processes $W_{1,t}$ and $W_{2,t}$.

When $\rho = 0$, the PDF of the detrended log-return $x \equiv \ln(S_\tau/S_0) - \mu\tau$ takes the semi-closed form (Drăgulescu and Yakovenko 2002)

$$P(x) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} e^{ikx + F(k)}$$

with

$$F(k) = \frac{\gamma^2 \theta}{\kappa^2} \tau - \frac{2\gamma\theta}{\kappa^2} \ln \left(\cosh \frac{\gamma\Omega\tau}{2} + \frac{\Omega^2 + 1}{2\Omega} \sinh \frac{\gamma\Omega\tau}{2} \right),$$

where the frequency is given by $\Omega \equiv \sqrt{1 + (k\kappa/\gamma)^2}$.

We fit GBM through maximum likelihood estimation. For the Heston model, we estimate parameters (Θ) by minimizing the mean-square deviation (Drăgulescu and Yakovenko 2002)

$$\Phi(\Theta) = \sum_{r_j, \tau} \left| \ln(\hat{P}_\tau(r_j)) - \ln(P_\tau(r_j; \Theta)) \right|^2$$

where $r_j \equiv R_j - \mu\tau$ is the detrended return, $P_\tau(r_j; \Theta)$ is the theoretical PDF for given parameters $\Theta = (\gamma, \theta, \kappa, \mu, \alpha)$, and $\hat{P}_\tau(r_j)$ is the empirical density obtained via (i) partitioning the r -axis into equally spaced bins (Scott 1979) and (ii) dividing the number of observations falling in each bin by the bin size Δr and the total number of observations.

Table 6. Parameter estimates of the Heston model

γ	θ	κ	μ	α
3.371×10^{-2}	1.335×10^{-4}	3.044×10^{-3}	1.301×10^{-4}	0.9718

The results in Table 6 show that $\alpha = 2\gamma\theta/\kappa^2 \approx 1$ and $\gamma\tau \ll 1$, which implies that the PDF of the Heston model converges to an exponential distribution, $P(x) \propto \exp(-|x|\sqrt{2/\theta\tau})$ (Silva *et al.* 2004).

We thus fit the Heston model with an exponential distribution by means of the maximum likelihood estimation and present estimated parameters for the different holding periods in Table 3.